Introduction to Mathematics and Modeling

lecture 3

Differentiation

UNIVERSITY OF TWENTE.

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lecture	:	3
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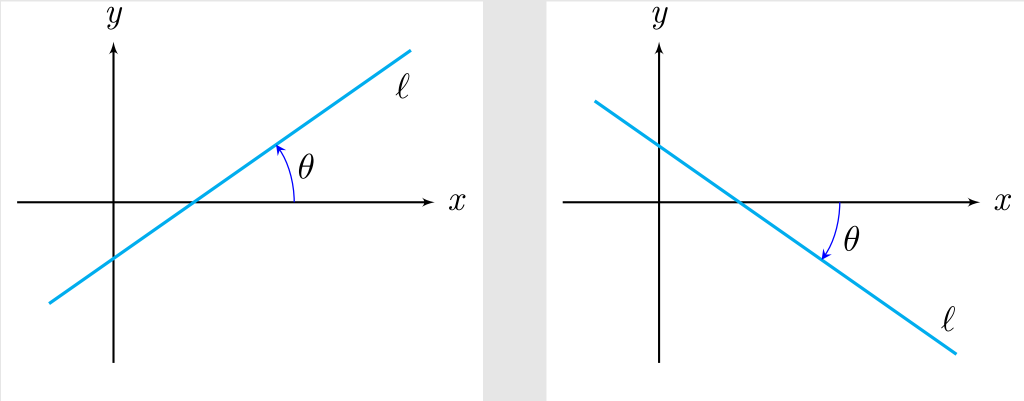
slides	:	25

This week

intro

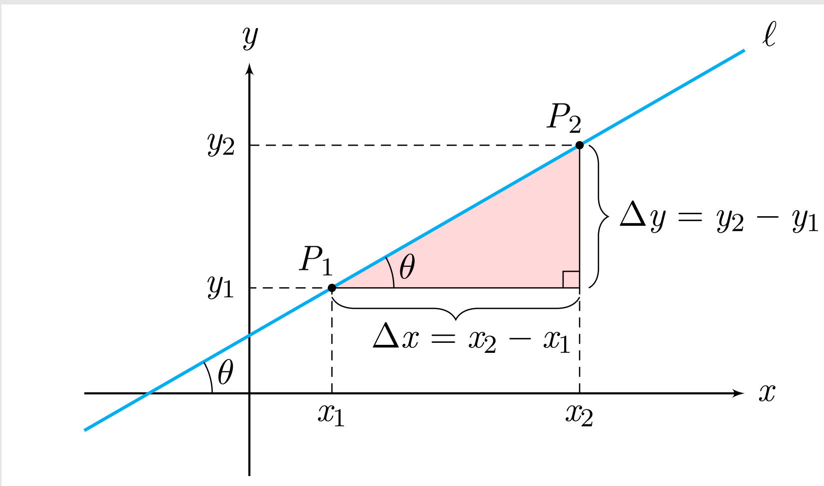
1. Directives concerning the MLP test
2. Section 3.1: tangents and the derivative at a point
3. Section 3.2: the derivative as a function
4. Section 3.4: velocity

Inclination



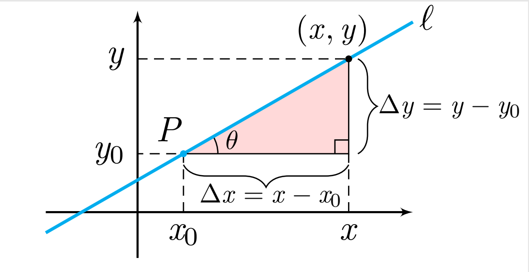
- The **(angle of) inclination** is the angle θ that ℓ makes with the horizontal axis.
- The angle is measured from the positive x-axis to ℓ.
- Turning counterclockwise means $\theta > 0$.
- Turning clockwise means $\theta < 0$.

The slope of a line



- The **slope of** ℓ is $\tan\theta = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- This holds for *every* choice $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, as long as $P_1 \neq P_2$.

Equation of a line through a point with given slope



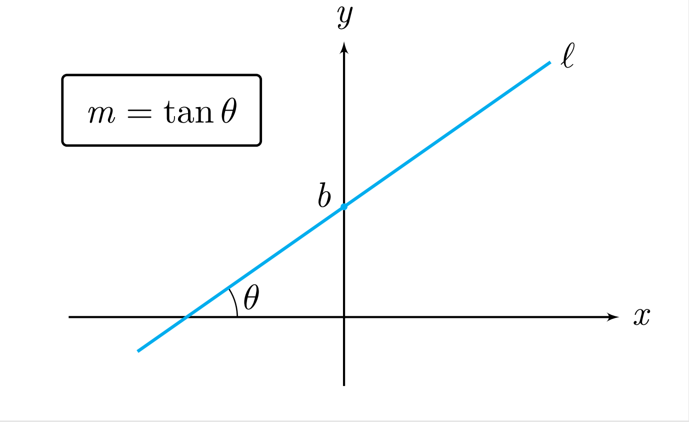
- Let ℓ be the line through $P = (x_0, y_0)$ with slope m, then for every point $(x, y) \neq P$ on ℓ we have

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} && \times(x - x_0) \\ y - y_0 &= m(x - x_0) && +y_0 \\ y &= m(x - x_0) + y_0. \end{aligned}$$

- The **equation of the line through** P **and with slope** m is

$$\boxed{y = m(x - x_0) + y_0}$$

Equation of a line with given slope and y-intercept



- Let ℓ be the line through with slope m and with y-intercept b, then ℓ passes through $(0, b)$.
- The equation of ℓ is $y = m(x - 0) + b$, simplified:

$$\boxed{y = mx + b}$$

The derivative of a function

We define the **derivative of f at x_0** as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

The number $f'(x_0)$ can be interpreted as:

- the slope of the graph of $y = f(x)$ at the point $(x_0, f(x_0))$;
- the slope of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$;
- the rate of change of $f(x)$ at the point x_0.

Differentiation - Secant.nb

Example

Example

Calculate the derivative of $f(x) = x^2$ at 1 with the definition.

h	$1+h$	$f(1)$	$f(1+h)$	$f(1+h)-f(1)$	$\frac{f(1+h)-f(1)}{h}$
1	2	1	4	3	3
0.5	1.5	1	2.25	1.25	2.5
0.25	1.25	1	1.5625	0.5625	2.25
0.01	1.01	1	1.0201	0.0201	2.01
0.001	1.001	1	1.002001	0.002001	2.001

This suggests: when h approaches 0, then $\frac{f(1+h)-f(1)}{h}$ approaches 2.

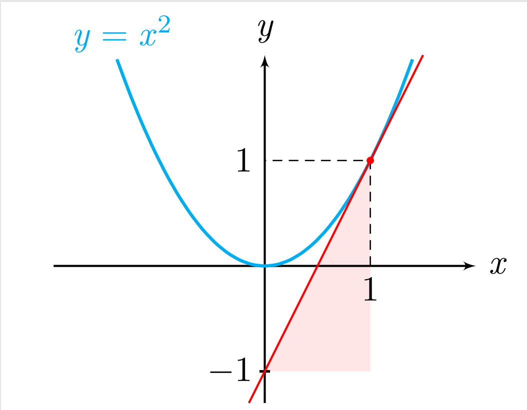
Example

Example

Calculate the derivative of $f(x) = x^2$ at 1 with the definition.

$$\frac{f(1+h) - f(1)}{h} =$$

Example



- The tangent line has slope $f'(1) = 2$ and passes through $(1, f(1)) = (1, 1)$.
- Hence the tangent line is described by the equation

$$\boxed{y = 2x - 1}$$

Example

Calculate the derivative of $f(x) = x^2$ at a with the definition.

$$\frac{f(a+h)-f(a)}{h} =$$

Example

Calculate the derivative of $f(x) = \sqrt{x}$ at a with the definition.

$$\frac{f(a+h)-f(a)}{h} =$$

Example 2.7

Example

Calculate the derivative of $f(x) = \dfrac{1}{x}$ *at* $a \neq 0$ *with the definition.*

$$\frac{f(a+h)-f(a)}{h} =$$

The derivative as a function 3.1

Definition

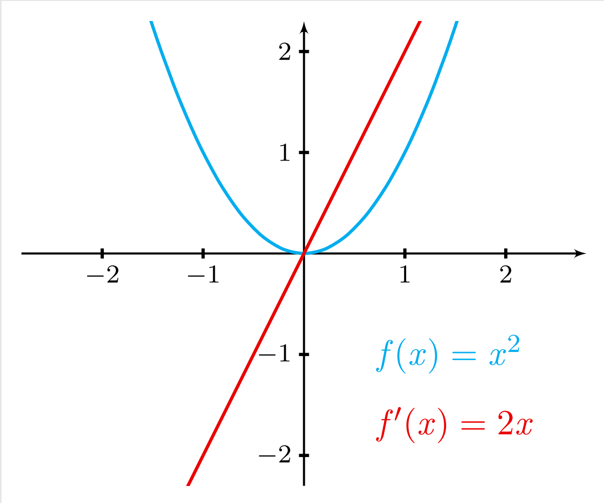
The **derivative of the function** f *is the function* f' *whose value at* x *is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

- The function f is **differentiable at** x if $f'(x)$ exists.
- The process of calculating f' is called **differentiation**.
- Alternative notations for the derivative are

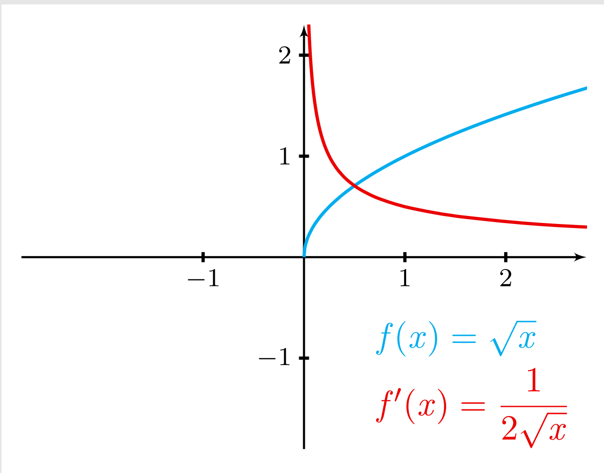
$$\frac{df}{dx} \quad \text{and} \quad \frac{d}{dx}f(x).$$

Example



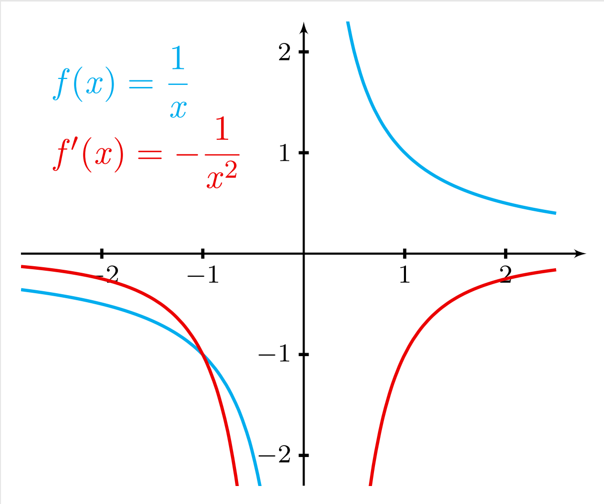
- On slide 10: the derivative of f at a is $f'(a) = 2a$.
- Replace a by x: the derivative of f is the function $f'(x) = 2x$.

Example



- On slide 11: $f'(a) = \dfrac{1}{2\sqrt{a}}$.
- Replace a by x: $\boxed{f'(x) = \dfrac{1}{2\sqrt{x}} \quad (x > 0)}$

Example: the derivative of $f(x) = 1/x$



- On slide 12: the derivative of f at a is $f'(a) = -\dfrac{1}{a^2}$.
- Replace a by x: $\boxed{f'(x) = -\dfrac{1}{x^2} \quad (x \neq 0)}$

Powers of x

Theorem

For all real numbers α we have $\boxed{\dfrac{d}{dx}(x^\alpha) = \alpha\, x^{\alpha-1}}$

Check:

- Let $\alpha = \frac{1}{2}$, then

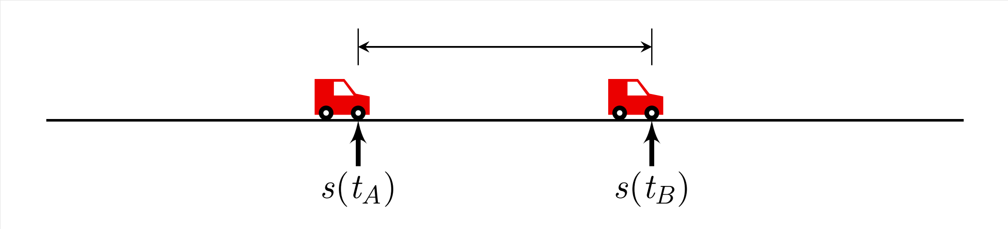
$$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) =$$

- Let $\alpha = -1$, then

$$\frac{d}{dx}\left(x^{-1}\right) =$$

Average velocity

Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

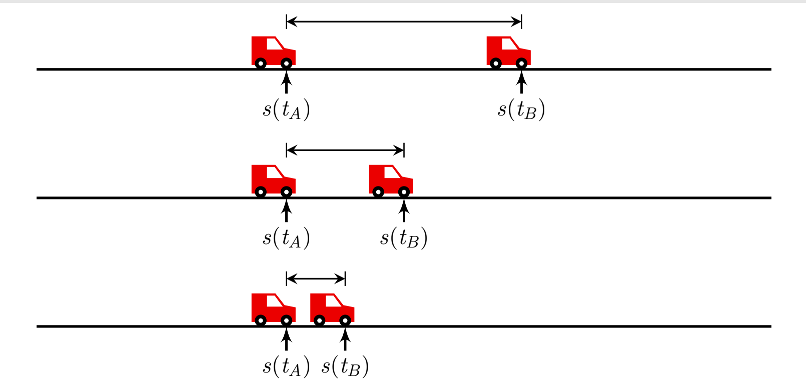


- If the object moves from $s(t_A)$ to $s(t_B)$, the **displacement** is $s(t_B) - s(t_A)$.
- The **average velocity over the interval** (t_A, t_B) is the displacement per elapsed time, and is equal to

$$\boxed{\frac{s(t_B) - s(t_A)}{t_B - t_A}}$$

Velocity

Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.



- The **velocity at time** t_A is the limit of the average velocity over the interval (t_A, t_B) where t_B approaches t_A:

$$v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}.$$

Velocity

$$v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}.$$

Define $h = t_B - t_A$, then

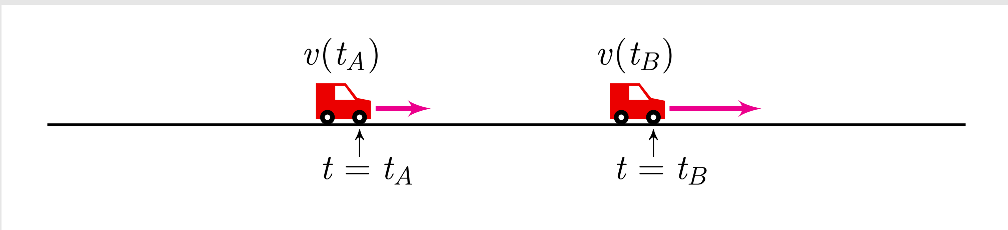
- $t_B = t_A + h$ and
- "$t_B \to t_A$" is equivalent to "$h \to 0$".

$$\begin{aligned} v(t_A) &= \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A} \\ &= \lim_{h \to 0} \frac{s(t_A + h) - s(t_A)}{h} = s'(t_A). \end{aligned}$$

Velocity is the derivative of displacement

Average acceleration

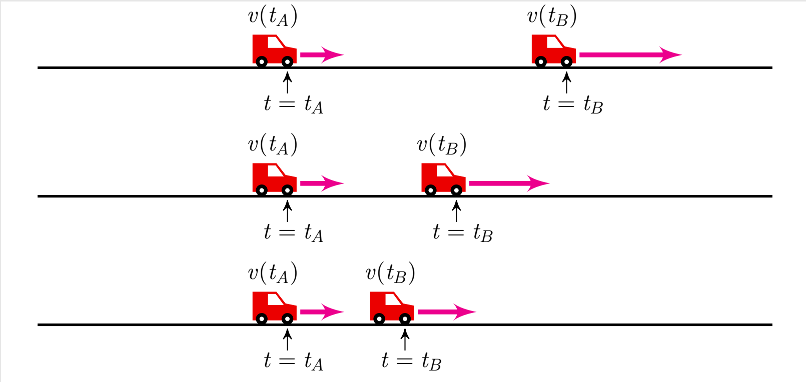
Consider a moving object and assume that we know the velocity as a function of time $v(t)$.



- The **average accelaration over the interval** (t_A, t_B) is the change in velocity per elapsed time, and is equal to

$$\frac{v(t_B) - v(t_A)}{t_B - t_A}$$

Acceleration 4.5



- The **acceleration at time** t_A is the limit of the average acceleration over the interval (t_A, t_B) where t_B approaches t_A:

$$a(t_A) = \lim_{t_B \to t_A} \frac{v(t_B) - v(t_A)}{t_B - t_A} = v'(t_A).$$

Acceleration is the derivative of velocity

Higher order derivartives 4.6

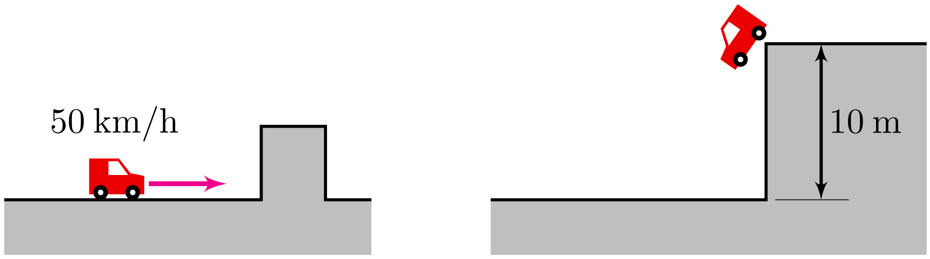
Definition

*Let n be a non-negative integer. The n-**th derivative of** f is denoted as $f^{(n)}$ or $\dfrac{d^n f}{dx^n}$, and is defined as*

$$f^{(n)}(x) = \begin{cases} f(x) & \text{if } n = 0, \\ f'(x) & \text{if } n = 1, \\ \dfrac{d}{dx}\left(f^{(n-1)}(x)\right) & \text{otherwise.} \end{cases}$$

- The second derivative is denoted as f'' and not as $f^{(2)}$.
- Acceleration is the second derivative of displacement: $a(t) = s''(t)$.

Which is better: falling or chrashing?



Capacitors

Physical principles

(1) *In a capacitor, the charge Q on the plates is proportional to the voltage V over the plates: hence $Q = CV$, where C is the* **capacity**.

(2) *The current through a lead is the amount of charge per second flowing through the lead.*